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THE OPTIMIZATION OF MULTISTAGE ROCKETS

WITH RESPECT TO COST

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ABSTRACT

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A mathematical method is presented for the computation of the stage weights of multistage rocket systems to give minimum rocket-system cost for a given mission. The mission is assumed presentable in terms of payload weight and a velocity requirement. The rocket system can have any number of stages of various types of rockets. Besides taking into account the usual factors of individual-stage specific impulse and structural weight fraction, the individual-stage specific cost or cost per pound is also included. The solution is presented in the form of a fairly concise set of formulas for the stage weight ratios. If the stage specific costs are dropped from the equations, the method gives the same results as the simpler concept of optimization with respect to weight. Thus both methods are included in one technique. Two examples are given which show that great differences in stage weight ratios result when optimizing with respect to weight or cost. The relative vehicle costs by the two methods differ by 25 percent in the first example and 300 percent in the second example.

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THE OPTIMIZATION OF MULTISTAGE ROCKETS

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INTRODUCTION

To accomplish space missions requiring the high velocities associated with escape from the earth, tandem-staged rocket systems are essential. With the current technology available for many types of rockets, the selection of an appropriate number of stages, the selection of the type of motor for each stage, and finally the determination of the correct weight of each of the stages are more complex tasks than ever before. The theme of the present paper is that the selection of the booster system and the optimum rocket sizes to be used should be based on the economics of a successful mission achievement. The substance of the paper is an optimization method to accomplish this result. The complex situation in which we find ourselves with respect to the wealth of possible motor types from which to choose may be better understood if a few remarks are given pertaining to the history of rocket motors and staged rocket systems.

The earliest recorded use of rockets occurred in 1232 A.D. when a son of Genghis Khan attacked Kaifeng, the capital of Honan province, China, with solid-propellant rockets. Such rockets were also used effectively by Indian soldiers against the British in the British Indian Campaign. As a result of this experience the British developed their own solid-propellant rockets which were used against Napoleon and against the Americans in the War of 1812. The use of rocket artillery gave way to the cannon for a time but was revived in World War II, being employed extensively by the Russians at Stalingrad and by the Allies in the Pacific and European theaters in single-stage ground-to-ground and air-to-ground applications. The NACA also employed two-stage solid-propellant rocket systems for transonic and supersonic aerodynamic tests at their Wallops Island launch base during the latter part of World War II.

Liquid-propellant rocket motors were introduced experimentally by R. H. Goddard in 1926 and in practice by the Germans in World War II in their V-2 application that used alcohol and liquid oxygen as propellants. Subsequently the Russians and Americans developed liquid-propellant IRBM's and ICBM's that utilized petroleum fuel.

Needing higher speeds for aerodynamic testing, the NACA developed three-, four-, five-, and six-stage solid-propellant rocket vehicles and demonstrated the good reliability of these multistage vehicles. Next,

the U.S. military services, with the cooperation of American industry, developed the three-stage Polaris, the three-stage Minuteman, and the two-stage Pershing. These vehicles gave the United States a complete second generation of solid-propellant ballistic missiles. Concurrently the NASA developed the four-stage solid-propellant rocket orbital Scout system.

The advent of the satellite witnessed the early use of liquid-propellant lower stages with solid-propellant upper stages. Recently, upper stages with more exotic storable and restartable liquid propellants have been utilized, as in the Atlas-Agena system. Hydrogen-fuel upper stages are under current development to obtain the highest possible specific impulse, as in NASA's Saturn rocket vehicles. Extremely large solid-propellant rockets are under consideration for use in alternate systems, mainly as replacements for first stages.

This short history illustrates the ever growing range of possible rocket types to choose from and the ever growing complexities of rocket systems. Note that while earlier space applications used solid-propellant rockets for upper stages, current thinking would employ them as the first stage. Thus the permutation, or order, is an additional consideration in multistaged systems.

Optimization studies can be expected to strongly aid or guide the choice of motor type for each stage and to dictate the appropriate size or weight for each stage. The older optimization methods had as their objective the optimization of performance or, what amounts to the same thing, minimization of total weight for a given performance as dictated by the mission. Numerous articles have been written giving mathematical methods for such optimization. The earliest methods took into account only the stage jet velocities, but later methods (refs. 1 to 3) also took into account the stage structural mass fractions, which is the fraction of inert (nonpropellant) weight in each stage. With the advent of space programs in which the rocket systems will cost millions or even billions of dollars, it seems very important that the objective of the optimization system be to minimize the cost. This is currently being done by machine programming methods wherein all possible parameters are varied systematically in what amounts to a "trial-and-error" procedure feasible only because of the high speeds of modern computing machines. The method herein presented is, by contrast, a mathematical method for the calculation of optimum staging ratios for minimum cost from given formulas. The labor required is slight, being the same as that required by the older methods which minimized weight.

The method applies to any number of stages and takes into account the stage jet velocities, stage mass fractions, and stage specific costs (dollars per pound). Stage specific costs are, in turn, determined by the type of rocket and the number of anticipated firings. The effect of reliability is also considered.

For systems like the Polaris or Minuteman where all stages are similar and where dollar costs per pound for each stage are not very different, optimization with respect to gross weight may be expected to approximate a minimum-cost vehicle. However, when stages with greatly different costs per pound are used, as when liquid-fuel stages are combined with the cheaper solid-fuel stages, optimization with respect to gross weight is misleading, as will be shown by two examples.

No mathematical optimization procedure for staged rockets can be expected to predict precise final results for either gross weight or cost, and such methods should not be expected to replace detailed design studies. However, optimization procedures can be expected to permit the selection of the best combination of rockets for the system and to orient design studies quickly. Furthermore, the effort expended in making a few optimization calculations based on a mathematical procedure is almost nothing compared with the value of such calculations in selecting systems and in orienting the laborious detailed weight and cost studies which must be done eventually for the system selected.

SYMBOLS

c	stage specific cost, dollars/lb
C	total rocket cost per firing, dollars
C _s	total rocket cost per successful firing, dollars
g	acceleration due to gravity, ft/sec
m _j	staging mass ratio, $\frac{W_j}{W_{j+1} + W_{j+2} + \dots + W_n + W_L}$
n	total number of stages in tandem
p	number of rockets in a stage
P	stage reliability, or probability of successful operation
r _j	jth stage mass ratio, $\frac{W_j + W_{j+1} + \dots + W_n + W_L}{\beta_j W_j + W_{j+1} + \dots + W_n + W_L}$
v	jet velocity, ft/sec
V _i	ideal velocity at nth stage burnout

W stage weight, includes interstage structure and controls, lb
 W_0 gross weight, lb
 W_L payload weight, lb
 β stage structural weight fraction
 λ Lagrange multiplier
 ϕ restraint function

Subscripts:

1,2,3,...j,...n 1st, 2nd, 3rd, ... jth, ... nth stages

The stage weights W must include the weight of interstage connections and the weight of controls. The stage structural weight fractions β must include these weights also. The structural weight fraction is defined as the ratio of the stage weight at stage burnout to the stage weight loaded with propellant.

ANALYSIS

The factors assumed known for rockets of a given type are: structural weight fractions β_j , effective jet velocities v_j , and stage cost per pound c_j . The quantities to be found are: stage mass ratios r_j and stage weights W_j for minimum mission propulsion cost. In this analysis it is assumed that each β_j and c_j is constant within the limits of weight variation considered for a given stage.

The cost of an n-stage rocket system is given by

$$\left. \begin{aligned} C &= c_1 W_1 + c_2 W_2 + c_3 W_3 + \dots + c_n W_n \\ &= \sum_{j=1}^n c_j W_j \end{aligned} \right\} \quad (1)$$

To obtain the cost per successful propulsion mission, we must divide the cost of the vehicle and payload by the product of the stage reliabilities. This cost includes the payload cost.

$$C_s = \frac{c_1 W_1 + c_2 W_2 + \dots + c_n W_n + c_L W_L}{P_1 P_2 \dots P_n} \quad (1a)$$

The stage reliability depends on the reliability of the rocket or rockets comprising the stage and on the number of such rockets clustered together. Assuming that the number of motors clustered is the same and that the stage reliability is constant, it is easy to show that the same stage weights result irrespective of whether equation (1) or (1a) is used in the derivation of the optimization formulas.

The cost is to be minimized subject to the condition that a given payload is accelerated to a given velocity, which of course depends on the assigned mission. The usual optimization procedure is to assign an ideal (or gravity-free drag-free) velocity V_1 that experience or trajectory computation indicates is the equivalent of the given mission actual burnout velocity. The speed equation is then extremely simple:

$$V_1 = v_1 \log r_1 + v_2 \log r_2 + \dots + v_n \log r_n \quad (2)$$

The Lagrangian multiplier method is convenient for this problem. The Lagrangian equation for this case is simply

$$L = C + \lambda \phi \quad (3)$$

where C is the sum to be minimized and ϕ is the velocity restraint obtained from equation (2) and given by

$$\phi = V_1 - v_1 \log r_1 - v_2 \log r_2 - \dots - v_n \log r_n \quad (4)$$

Substituting equations (1) and (4) into equation (3) gives

$$L = c_1 W_1 + c_2 W_2 + \dots + c_n W_n + \lambda (V_1 - v_1 \log r_1 - v_2 \log r_2 - \dots - v_n \log r_n) \quad (5)$$

Taking the derivatives of the Lagrangian equation (5) with respect to the weights W_j and equating to zero yields

$$\frac{\partial L}{\partial W_j} = c_j + \lambda \frac{\partial \phi}{\partial W_j} = 0$$

or

$$\frac{\partial \phi}{\partial W_j} = - \frac{c_j}{\lambda} = \text{Constant} \quad (6)$$

Note that $\frac{\partial \phi}{\partial W_j}$ can be expressed as

$$\frac{\partial \phi}{\partial W_j} = \frac{\partial \phi}{\partial r_1} \cdot \frac{\partial r_1}{\partial W_j} + \frac{\partial \phi}{\partial r_2} \cdot \frac{\partial r_2}{\partial W_j} + \dots + \frac{\partial \phi}{\partial r_j} \cdot \frac{\partial r_j}{\partial W_j} \quad (7)$$

Getting $\frac{\partial \phi}{\partial r_1}$, $\frac{\partial \phi}{\partial r_2}$, etc., of equation (7) from equation (4) and combining equation (7) with equation (6) yields

$$\frac{c_j}{\lambda} = \frac{v_1}{r_1} \cdot \frac{\partial r_1}{\partial W_j} + \frac{v_2}{r_2} \cdot \frac{\partial r_2}{\partial W_j} + \dots + \frac{v_j}{r_j} \cdot \frac{\partial r_j}{\partial W_j} \quad (8)$$

where j takes all values from 1 to n . The expressions for $\frac{\partial r_1}{\partial W_j}$, $\frac{\partial r_2}{\partial W_j}$, etc., can be obtained from the definition of the staging ratios

$$\left. \begin{aligned} r_1 &= \frac{W_1 + W_2 + \dots + W_n + W_L}{\beta_1 W_1 + W_2 + \dots + W_n + W_L} \\ r_2 &= \frac{W_2 + \dots + W_n + W_L}{\beta_2 W_2 + \dots + W_n + W_L}, \text{ etc.} \end{aligned} \right\} \quad (9)$$

Substituting the values of $\frac{\partial r_1}{\partial W_j}$, $\frac{\partial r_2}{\partial W_j}$, etc., in equation (8) and using the following identities:

$$\left. \begin{aligned} m_1 &\equiv \frac{W_0}{W_0 - W_1} \\ m_1 m_2 &\equiv \frac{W_0}{W_0 - W_1 - W_2} \\ m_1 m_2 m_3 &\equiv \frac{W_0}{W_0 - W_1 - W_2 - W_3}, \text{ etc.} \end{aligned} \right\} \quad (10)$$

to simplify equation (8) yields the following set of equations for calculating the stage mass ratios r_j :

$$\frac{W_0}{\lambda} = \frac{v_1}{c_1} (1 - \beta_1 r_1) \quad (11)$$

$$r_2 \beta_2 = 1 - \frac{c_2 (W_0/\lambda) + v_1 (r_1 - 1)}{v_2 m_1} \quad (12)$$

$$r_3 \beta_3 = 1 - \frac{c_3 (W_0/\lambda) + v_1 (r_1 - 1) + v_2 m_1 (r_2 - 1)}{v_3 m_1 m_2} \quad (13)$$

$$r_4 \beta_4 = 1 - \frac{c_4 (W_0/\lambda) + v_1 (r_1 - 1) + v_2 m_1 (r_2 - 1) + v_3 m_1 m_2 (r_3 - 1)}{v_4 m_1 m_2 m_3} \quad (14)$$

$$r_5 \beta_5 = 1 - \frac{c_5 (W_0/\lambda) + v_1 (r_1 - 1) + v_2 m_1 (r_2 - 1) + v_3 m_1 m_2 (r_3 - 1) + v_4 m_1 m_2 m_3 (r_4 - 1)}{v_5 m_1 m_2 m_3 m_4} \quad (15)$$

Formulas for greater numbers of stages than five can now be written by inspection. All staging ratios can be calculated by the formula

$$m_j = \frac{r_j(1 - \beta_j)}{1 - \beta_j r_j} \quad (16)$$

for example

$$m_1 = \frac{r_1(1 - \beta_1)}{1 - \beta_1 r_1}$$

The procedure to obtain the stage mass ratios is to take a trial value of r_1 and calculate W_0/λ from equation (11), r_2 from equation (12), r_3 from equation (13), r_4 from equation (14), and r_5 from equation (15), etc., according to the number of stages. With each step, the values of m required are obtained from equation (16). Substitute the trial set of r 's in equation (2) to calculate the ideal velocity. Revise r_1 and recalculate the other r 's as required to give the desired ideal velocity. It should be noted that each set of r 's defines a minimum cost set of rockets for the corresponding values of V_1 as given by equation (2).

Once the stage mass ratios have been determined, calculate the stage weights from the known values of m_j , starting with the last stage.

$$\left. \begin{aligned} W_n &= (m_n - 1)W_L \\ W_{n-1} &= m_n(m_{n-1} - 1)W_L \\ W_{n-2} &= m_n m_{n-1}(m_{n-2} - 1)W_L, \text{ etc.} \end{aligned} \right\} \quad (17)$$

Then, the costs are obtained from equations (1) and (1a).

In order to calculate the minimum weight set of rockets for a given ideal velocity performance, let the values of specific cost c_1 , c_2 , c_3 , c_4 , etc., in equations (11) through (15) equal unity. The c 's then disappear from these equations. Then follow exactly the same procedure outlined above to obtain minimum cost using equations (11) through (17).

EXAMPLE 1

Let it be required to send a payload of 16,600 pounds to an escape speed of 36,000 ft/sec. Trajectory studies available indicate an ideal velocity of 40,000 ft/sec is required. It is desired to investigate a system composed of a clustered solid-fuel first stage, together with clustered hydrogen-fuel second and third stages. The costs per pound for the different stages, together with the values of β and v , are listed in table I. From these data and from equations (11), (12), (13), and (2), the stage mass ratios for minimum cost were determined by the procedure described. These ratios and the stage weights and costs are also given in table I, case A.

In order to minimize the weight, the same procedure and equations are used except that the cost per pound is taken to be the same for all stages; that is, $c_1 = c_2 = c_3 = 1$. The c 's then disappear from equations (11), (12), and (13). Using the same assumptions as in the first part of Example 1, the resulting stage weights and costs were computed and are given in table I, case B. These results are identical to those obtained by other methods, such as reference 1.

The total cost per shot and weight per shot for both cases are shown in the last column of table I. For the minimum-cost solution, it is seen that the cost is only 0.76 of the cost for the minimum-weight solution, although the weight is 1.21 that of the minimum-weight solution. To obtain the cost per successful shot, the stage reliabilities must be estimated. This will be demonstrated in example 2.

EXAMPLE 2

The second example will be a four-stage vehicle with the same requirement; namely, a payload of 16,600 pounds and an ideal velocity of 40,000 ft/sec. Consider a solid-fuel first stage with three hydrogen-fuel upper stages: Case A, minimize cost; Case B, minimize weight.

The costs per pound for the different stages, together with the values of β and v , are listed in table II. Making use of the same procedures as before with equations (11), (12), (13), (14), and (2), the stage mass ratios for minimum cost were computed. These together with the stage weights and costs are shown in table II, case A.

Minimization of weight is accomplished by putting

$$c_1 = c_2 = c_3 = c_4 = 1$$

in equations (11) to (14) and results in a value of r_1 less than 1. This indicates a negative first stage, which of course is impossible. Let $r_1 = 1$, which means that the rocket system would not have a solid first stage but would consist of a three-stage hydrogen system, having the stage weights and stage costs per shot shown in table II, case B.

Since we are now comparing rockets with different numbers and types of components and consequently different reliabilities, the rocket cost comparison must be based on the cost per successful rocket mission. For simplicity in the following, the control reliability is not considered. For our purpose let us assume the number of motors in parallel shown in table II. We also assume that the reliability of each hydrogen motor is 0.96 and that the reliability of each solid motor is 0.99. Each stage reliability can be estimated by raising the motor reliability to a power equal to the number of stages in parallel. Thus for case A:

$$P_1 = (0.99)^3 = 0.9703$$

$$P_2 = (0.96)^4 = 0.8493$$

$$P_3 = (0.96)^2 = 0.9216$$

$$P_4 = (0.96)^1 = 0.96$$

$$P_1 P_2 P_3 P_4 = 0.7291$$

From equation (1a), allowing a payload cost of \$10 million:

$$C_s = \frac{23,783,000}{0.7291} = \$32,620,000$$

per successful shot, including payload.

For case B (continuing with the fictitious first stage):

$$P_1 = 1$$

$$P_2 = (0.96)^{16} = 0.5204$$

$$P_3 = (0.96)^4 = 0.8493$$

$$P_4 = (0.96)^1 = 0.96$$

$$P_1 P_2 P_3 P_4 = 0.4243$$

and

$$C_s = \frac{33,981,000}{0.4243} = \$80,087,000$$

per successful shot, including payloads. This value emphasizes the importance of reliability.

In example 2 there is about \$9 million difference in cost per shot fired but there is about \$47 million per successful shot difference between optimizing with respect to cost and optimizing with respect to performance.

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TABLE I

(Example 1)

THREE-STAGE VEHICLE: ONE SOLID AND TWO HYDROGEN-LOX

	<u>Stage 1</u>	<u>Stage 2</u>	<u>Stage 3</u>	<u>Total weight and Cost/Shot</u>
c, \$/lb . . .	6	26	26	
β	0.14	0.14	0.14	
I, sec	254	420	420	
v, ft/sec . .	8,179	13,524	13,524	
Case A: { r	3.166	3.098	3.098	
Minimum { W, lb	1,429,000	289,300	61,500	1,779,800
cost { cW, \$	8,576,000	7,523,000	1,599,000	17,698,000
Case B: { r	1.630	3.809	3.809	
Minimum { W, lb	667,300	701,000	99,900	1,468,200
weight { cW, \$	4,004,000	18,226,000	2,597,000	24,827,000

TABLE II

(Example 2)

FOUR-STAGE VEHICLE: ONE SOLID AND THREE HYDROGEN-LOX

[Payload, 16,600 lb; cost, \$10,000,000; $V_1 = 40,000$ ft/sec]

	Stage 1	Stage 2	Stage 3	Stage 4
c, \$/lb	6	26	26	26
β	0.14	0.14	0.14	0.14
v, ft/sec . . .	8,179	13,524	13,524	13,524
r	2.751	2.1858	2.1858	2.1858
W, lb	939,480	208,100	76,830	28,360
cW, \$	5,637,000	5,411,000	1,998,000	737,000
p	3	4	2	1
P	0.9703	0.8493	0.9216	0.960

Case A:
Minimum
cost

$$\sum_1^4 W_j = 1,252,770 \text{ lb}$$

$$\sum_1^4 c_j W_j = \$13,783,000/\text{shot}$$

$$C_B = \frac{23,783,000}{0.7291} = \$32,620,000/\text{successful shot}$$

Case B:
Minimum
weight

v, ft/sec . . .	None	^a 12,490	13,524	13,524
r	1	2.513	2.867	2.867
W, lb	0	657,360	213,240	51,770
cW, \$	0	17,091,000	5,544,000	1,346,000
p	0	16	4	1
P	1	0.5204	0.8493	0.96

$$\sum_2^4 W_j = 922,370 \text{ lb}$$

$$\sum_2^4 c_j W_j = \$23,981,000/\text{shot}$$

$$C_B = \frac{33,981,000}{0.4243} = \$80,087,000/\text{successful shot}$$

^aAdjusted for low-altitude operation.